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Computer Science

Combinatorial Morse Complexes

Abstract

Morse theory uncovers the topological structure of a shape based on the critical points of a real valued function over the shape. The Morse-Smale complex cuts a surface into a set of quadrilateral patches suitable for shape matching or reparameterization. We present a simple algorithm for the computation of Morse-Smale complexes on 2-manifold polygonal meshes. Our algorithm is based on discrete Morse theory, a combinatorial adaptation of classical Morse theory to cell complexes. Discrete Morse theory avoids the complexities of multi-saddles and provides a simple, robust method for simplification of the complex.

Professor Ken Joy
Thesis Committee Chair

Combinatorial Morse Complexes

By

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THESIS

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Combinatorial Morse Complexes

To My Family

Acknowledgments

Many thanks to my advisor, Ken Joy, for being the most friendly advisor one could ask for. I would also like to thank Valerio Pascucci (Lawrence Livermore National Laboratory) for inspiring this work, and Bernd Hamann (UC Davis) for his very prompt review of this thesis. I am indebted to Lawrence Livermore National Laboratory for providing me with three great summer internships (one as an undergraduate), which facilitated this work. I would also like to thank Anna Tikhonova for proof-reading and totally upgrading this thesis. Lastly, many thanks to my parents for their continual support.

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Abstract

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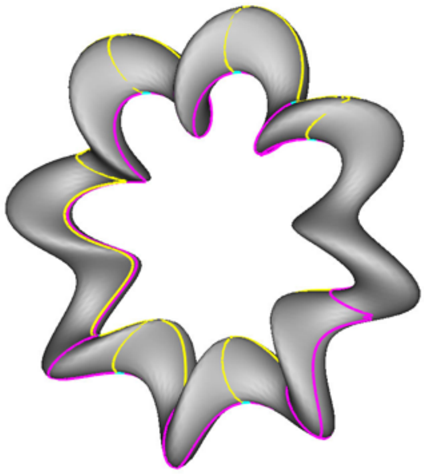
$X * Y$	join of X and Y
α, β	simplices
$\alpha + \beta$	union of simplices
v	vertex
$\alpha^\star$	dual of α
K, L	simplicial complexes
K_p	p-skeleton
M_p	critical p-simplices
$\beta < \alpha$	β is a facet of α
$St(v)$	star of v (13)
$St_-(v)$	lower star of v (17)
$Lk(v)$	link of v (14)
$Lk_-(v)$	lower link of v (18)
$\partial\alpha$	boundary of α
$\partial^{-1}\alpha$	co-boundary of α
$K \searrow L$	K contracts to L
V	discrete gradient field
Φ	discrete time flow
$< \alpha, \beta > \neq 0$	α incident on β
$\bar{u}$	closure of u
$\circledcirc$	minimum
$\odot$	maximum
$\oplus$	saddle

Table 0.1: Table of Notation

Chapter 1

Introduction



Scientific visualization and geometric modeling often present us with the problem of analyzing a real-valued function on a manifold (surface or volume). This function might originate from experimental measurements, such as in medical imaging or computational fluid dynamics, or from a function of the underlying space, such as curvature or the z-coordinate. Regardless of the source of our function, one important class of properties of the function we would like to know are its critical points and how they relate to each other. Critical points represent features, such as the peak of a mountain, the center of a particle or the intersection of two evolving surfaces. Relationships between critical points tell us, for example, the structure of a molecule, or tell us how a mountain peak is connected to passes and valleys, the structure of a mountain range. These relationships are *topological* properties.

This thesis presents and discusses new algorithms for the topological analysis of scalar functions, the analysis of critical points and the relationships between them.

Our Tool: Morse Theory

Morse theory provides an elegant relationship between differential geometry and topology. It relates the critical points of a smooth real-valued function defined on a manifold with the global topology of the manifold. Given a manifold M and a smooth function $f : M \to \mathbb{R}$ we can look at the critical points of f to find topological properties of M. For example, the genus, g, of a manifold without boundary describes the number of "holes" there are in the surface [1]. A sphere is genus 0, a torus 1, and a double torus 2. The genus of a connected manifold mesh is given by the Euler-Poincaré formula.

$$2 - 2g = \text{vertices} - \text{edges} + \text{faces}$$

We can also calculate the genus using the critical points of f as follows.

$$2 - 2g = \text{minima} - \text{saddles} + \text{maxima}$$

Morse theory also enables us to partition the manifold into regions of uniform gradient flow. This concept, the Morse-Smale complex, has seen numerous recent applications, from mesh reparameterization [5], to molecular shape analysis [3]. The efficient and practical computation of the Morse-Smale complex is the topic of this thesis.

Example

To understand the basics of Morse theory, we begin with a simple and common example: a vertically oriented torus and a height function, f (see Figure 1.1). Let M denote the 2-manifold surface of the vertically oriented torus, and let $f(p)$ denote the height of a point $p \in M$. Thus point a, at the bottom of the torus, is the global minimum of f and point d, at the top of the torus, is the global maximum of f.

Let M_s be the set of all points $x \in M$ such that $f(x) \leq s$.

[1] Alternatively, the genus is the number of closed curves that can be drawn on the surface without disconnecting it.

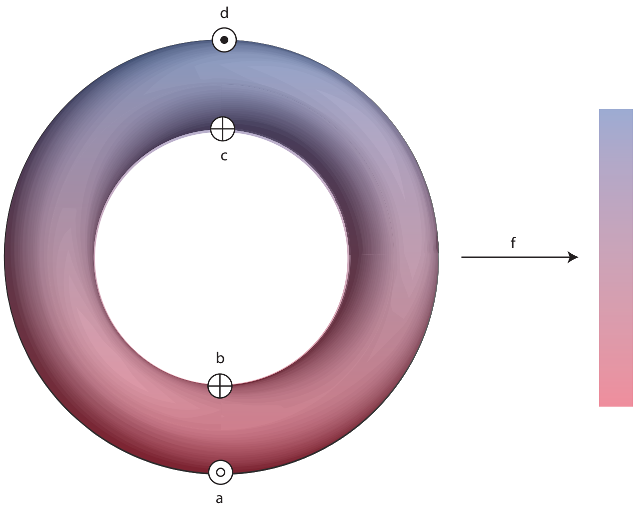


Figure 1.1: Torus. The function f is the z-coordinate. a is a minimum, b and c are saddles, and d is a maximum.

1. If $s < f(a)$ then M_s is empty.
2. If $f(a) < s < f(b)$ then M_s is homeomorphic[2] to a disk.
3. If $f(b) < s < f(c)$ then M_s is homeomorphic to a cylinder.
4. If $f(c) < s < f(d)$ then M_s is homeomorphic to a compact manifold of genus one with a circle as its boundary.
5. If $f(d) < s$ then M_s is the whole torus.

Thus, beginning with the global minimum a, as s assumes the value of each critical point, there is a change in topology of M_s. This illustrates the relationship between the critical points of f and the global topology of M: the topology of M_s is determined by the changes the critical points induce as s increases.

Now, consider the function in Figure 1.2. Here, a terrain refers to a height-map: $f : \mathbb{R}^2 \to \mathbb{R}$. Tthe topology of our terrain is just a disk. We will not deduce anything interesting about the manifold from the critical points, but they instead tell the relationship between features in the terrain (features of f).

[2] *Homeomorphic* means continuously deformable. Intuitively, two shapes are homeomorphic if one can be bent into the other without tearing, cutting, or gluing.

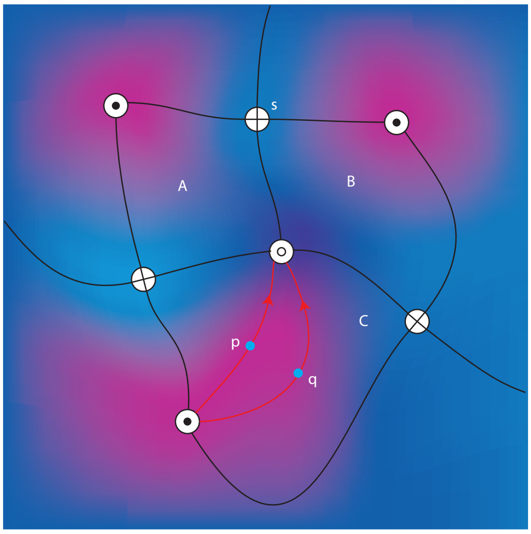


Figure 1.2: Terrain. A, B, and C are patches of the Morse-Smale complex. p and q lie on integral lines with the same endpoints, thus they both belong to C.

The saddle labeled s has two ways one can walk downwards fastest, and two ways one can walk upwards fastest. These paths are illustrated by the curves, called *integral lines*, coming out of s. In general, each integral line emanating from a saddle leads to an extremum, but never another saddle. Multiple integral lines can converge at an extremum. The integral lines are the boundaries of quadrilateral patches. The interiors of these quadrilateral patches are points where the integral lines have the same endpoints, as shown by point p in the figure. The collection of critical points, integral lines, and patches form the *Morse-Smale Complex*.

Combinatorial Approach

One could construct the Morse-Smale complex numerically by following the gradient from saddles. However, numerical instability would render this type of algorithm combinatorially incorrect. For example, when following the gradient to a local minimum, we could miss that minimum and reach a different minimum instead. The case of 3-manifolds (volume data-sets) compounds this problem, since we must trace lines between different types of saddles. Imagine rolling a ball so it will stop, precariously positioned, at the center of a saddle. Regions that are nearly flat would also cause additional instability, since their gradients are so small.

To ensure the combinatorial integrity of the Morse-Smale complex, we use combinatorial algorithms. The algorithms in this thesis are based on discrete Morse theory, an entirely combinatorial Morse theory [11]. Discrete Morse theory avoids the complexities of other combinatorial approaches [1] [25] and provides a simple, robust method for simplification of the complex.

Others have also constructed the Morse-Smale complex using discrete Morse theory. Cazals and Lewiner [3], in particular, give an $O(n \log n)$ algorithm for computing the Morse-Smale complex on polygon meshes. The algorithm in this thesis is linear in the number of vertices in the mesh.

Outline

This thesis is organized as follows: Chapter 1 introduces the mathematical concepts of classical Morse theory, focusing on what is relevant to this thesis. Chapter 2 discusses related work, including recent applications of discrete Morse theory. Chapter 3 introduces discrete Morse theory. Chapter 4 introduces a simple algorithm for the computation of discrete Morse-Smale complexes on polygon meshes. Chapter 5 presents the implementation of such an algorithm for volume data-sets.

Chapter 2

Morse Theory

Morse Theory connects *topology* and *differential geometry* in a simple and elegant way. This chapter provides a quick, accessible introduction to the concepts of classical Morse theory relevant to this thesis. Reference [21] provides proofs of the theorems and further details.

2.1 Critical Points

In this section, we formalize the notion of a critical point and the different types of critical points. First, we must define the *gradient* of a function, f. The gradient, a generalization of the derivative to multiple dimensions, is the direction of greatest increase of f.

Definition 1 (Gradient)**.** *Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, the* ***gradient*** *of f, is defined as:*

$$\nabla f = (\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_{n-1}})$$

A *critical point* is a point where the gradient vanishes. A more formal definition follows.

Definition 2 (Critical Point and Critical Value)**.** *A point x is called a* ***critical point*** *of a function f if $\nabla f(x) = 0$. $f(x)$ is called a* ***critical value*** *of f.*

In order to distinguish between different types of critical points we use the *Hessian* *matrix*, a matrix of second derivatives. This is a generalization of the second derivative test[1] from basic Calculus. The formal definition follows.

Definition 3 (Hessian Matrix)**.** *The* Hessian matrix *of* $f : \mathbb{R}^n \to \mathbb{R}$ *is*

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x_0} & \frac{\partial^2 f}{\partial x_0 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_0 \partial x_{n-1}} \\ \frac{\partial^2 f}{\partial x_1 \partial x_0} & \frac{\partial^2 f}{\partial^2 x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_{n-1} \partial x_0} & \frac{\partial^2 f}{\partial x_{n-1} \partial x_1} & \cdots & \frac{\partial^2 f}{\partial^2 x_{n-1}} \end{pmatrix}$$

The number of positive, negative, and zero eigenvalues of the Hessian determine the function's local behavior. For $f : \mathbb{R} \to \mathbb{R}$, the Hessian is simply the second derivative, and non-singularity is equivalent to being nonzero. In this case we apply the second derivative test to find the type of a critical point: minimum if $f''(x) > 0$, maximum if $f''(x) < 0$, or *degenerate* if $f''(x) = 0$. In general, the number of negative eigenvalues of the critical point's Hessian is called its *index*. If all eigenvalues are negative, the point is a maximum (index $n - 1$). If all are positive, the point is a minimum (index zero). The critical points of indices between zero and $n - 1$ are saddles. Functions in n dimensional space have $n - 2$ different types of saddles.

The notion of a degenerate critical point generalizes to arbitrary dimension as follows.

Definition 4 (Degenerate Critical Point)**.** *A critical point x of a function $f : \mathbb{R}^n \to \mathbb{R}$ is called* ***non-degenerate*** *if* $\det(H_f(x)) \neq 0$*. It is called* ***degenerate*** *otherwise.*

The next theorem shows that provided we have a non-degenerate critical point, f takes on a very simple structure in the neighborhood of the critical point. This fascinating fact is the basis of Morse theory.

Theorem 1 (Morse Lemma)**.** *Let p be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$. There exist local coordinates $(x_1, x_2, \ldots, x_n)$ in a neighborhood U of p such that*

$$f = f(p) - (x_1)^2 - \cdots - (x_k)^2 + (x_{k+1})^2 + \cdots + (x_n)^2$$

[1] The second derivative test is used to distinguish between minima and maxima of a function $\mathbb{R} \to \mathbb{R}$.

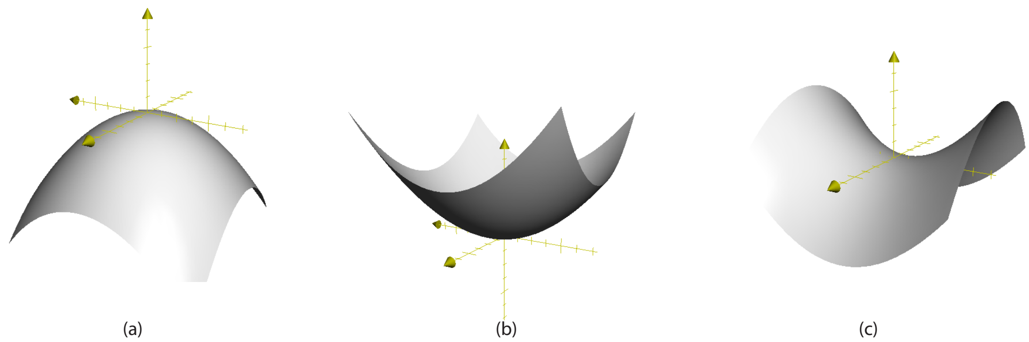


Figure 2.1: Standard forms of critical points for a 2-Manifold. (a) Maximum: $-x^2 - y^2$ (b) Minimum: $x^2 + y^2$ (c) Saddle: $x^2 - y^2$.

*throughout U. k is equal to the **index** of p.*

Thus, for functions $\mathbb{R}^2 \to \mathbb{R}$, the Morse Lemma implies that f will be of the form

$$f(x, y) = a + (Ax^2 + By^2)$$

This locally quadratic behavior is illustrated in Figure 2.1.

As a corollary, the Morse Lemma implies critical points are isolated, since there exists a neighborhood around the critical point p with no other critical points.

Finally, the following theorem provides a relationship between critical points and the topology of M.

Theorem 2 (Critical Point Theorem)**.** *Let $f : M \to R$ be a Morse function. Let c_i be the number of critical points of index i. The following relation holds:*

$$\sum_{k=0}^{d-1} (-1)^k c_k = \chi(M),$$

where $\chi(M)$ is the Euler characteristic *of the manifold M.*

The Euler characteristic is a topological invariant, a number that describes one aspect of a topological space's shape or structure. It is related to the *genus* by

$$\chi = 2 - 2g.$$

The genus is the number of handles on M, or, the number of closed curves that can be drawn on M without disconnecting it [16].

Also, the critical points dictate the topology of level sets. The following theorem, shows that two level sets are homeomorphic if there are no critical points between their iso-values.

Theorem 3. *Given an MS function f and $a, b \in \mathbb{R}$ such that there are no critical values in $[a, b]$, the level set $f^{-1}(a)$ is homeomorphic to $f^{-1}(b)$.*

2.2 Topological Spaces

A cell complex is a combinatorial structure for specifying a topological space. It is a generalization of a *simplicial complex* [16]. A cell α of dimension k is a topological space homeomorphic to the open ball $x \in \mathbb{R}^k : |x| < 1$. A *cell complex* is a collection of cells, defined as follows.

Definition 5 (Cell Complex)**.** *A (finite) cell complex (or CW-Complex), $\mathcal{K}$ is a set of cells (sets homeomorphic to balls), such that a n-cell has on its boundary $(n-1)$-cells which are also in the complex.*

For a cell, α, the operation of taking its boundary is denoted $\partial\alpha$. The geometric co-boundary operator, ∂^{-1}, is defined by $\partial^{-1}\alpha = \{\beta > \alpha\}$. We say α is a face of β (denoted $\alpha < \beta$) if $\alpha \in \partial\beta$.

Given a cell complex, $\mathcal{K}$, we call the *n-skeleton* of $\mathcal{K}$ the set $\mathcal{K}_n \subseteq \mathcal{K}$ of n-cells. The dimension of $\mathcal{K}$ is the highest value of n such that $\mathcal{K}_n$ is not empty. For convenience we will let $\emptyset < v$ for all vertices $v \in \mathcal{K}_0$, and we will define $\dim \emptyset = -1$. We also define $n_k =| \mathcal{K}_k |$ to be the number of k cells.

We can now define the dual complex, $\mathcal{K}^\star$, as follows.

Definition 6 (Dual Complex)**.** *For a cell complex $\mathcal{K}$ the* dual complex*, $\mathcal{K}^\star$ is a cell complex defined by the following rules:*

1. *For every $\alpha \in \mathcal{K}_p$ there is an $\alpha^\star \in \mathcal{K}^\star_{n-p}$.*

2. $\alpha < \beta$ *iff* $\alpha^\star > \beta^\star$*.*

See [16] for definitions and topological properties of cell complexes.

2.3 The Morse-Smale Complex

Definition 7 (Integral Line)**.** *An **integral line** $\gamma : \mathbb{R} \to M$ of a smooth function $f : M \to \mathbb{R}$ is a line with a tangent vector parallel to the gradient of f:*

$$\gamma'(t) = \nabla f(\gamma(t))$$

*The **origin** of γ is $\mathrm{org}(\gamma) = \lim_{t\to-\infty} \gamma(t)$. The **destination** of γ is $\mathrm{dest}(\gamma) = \lim_{t\to\infty} \gamma(t)$.*

Integral lines have the following properties:

1. Integral lines do not cross.

2. Integral lines cover M.

3. The origin and destination of an integral line are critical points of f.

Properties (1 and 2) are satisfied because there is a unique gradient at each point of M. Property (3) is satisfied because an integral line is monotonic. The unique integral line containing x is referred to as *the integral line of x*. All points on M can be classified by the origin and destination of their integral lines as follows.

Definition 8 (Stable and Unstable Manifolds)**.** *Given a Morse function $f : M \to \mathbb{R}$, the **stable manifold**, $S(p)$ of a critical point p is defined as the set of points with integral lines that end at p:*

$$S(p) = \{p\} \cup \{x \in M | x \in \mathrm{im}(\gamma) \textit{ and } \mathrm{dest}(\gamma) = p\}$$

*Symmetrically, the **unstable manifold**, $U(p)$ of p is defined as the set of points with integral lines that begin at p. Equivalently,*

$$U_f(p) = S_{-f}(p) \tag{2.1}$$

The classification of points on M by stable and unstable manifolds form a cell complex, defined as follows. This cell complex is called the *Morse complex.*

Definition 9 (Morse Complex)**.** *Given a Morse function* $f : M \to \mathbb{R}$*, the cell complex of stable manifolds is called the* ***Morse complex****.*

Definition 10 (Morse-Smale (MS) Function)**.** *A Morse-Smale function is a Morse function where no integral line has saddles at both its origin and destination.*

Definition 11 (Morse-Smale Complex)**.** *Given a MS function* f*, the* ***Morse-Smale complex (MS complex)*** *is the intersection between the Morse complex of* f *and* $-f$*.*

2.4 Topological Simplification

Morse complexes are related to the underlying topology of the space, but only in a weak manner. A Morse function only provides an upper bound on the topological complexity. Thus, a complicated Morse function does not imply that the topology is nontrivial. One could easily construct a Morse function with many critical points on a simple manifold such as a sphere. However, there are Morse functions with minimal number of criticalities. This minimum corresponds to the Betti number of the space.

Finding the topology of the underlying space is not the only motivation for simplification of the MS complex. Usually, in real-world data, there are many extraneous critical points which either correspond to noise or small topological features. In either case, we would like to simplify the MS-complex so we only see the features we care about.

Simplification of a MS-complex is carried out by contracting pairs of critical points. In order to know which pairs to contract, we use topological *persistence* [10], the standard measure of the significance of a topological feature, defined as follows.

Definition 12 (Persistence)**.** *Let* a *and* b *be critical points connected by a 1-cell of the MS complex. The* ***persistence*** *of the topological feature* (a, b) *is the difference in function value of the critical points:* $p = |f(a) - f(b)|$*.*

Intuitively, persistence measures how much f would need to be changed in order to remove a feature.

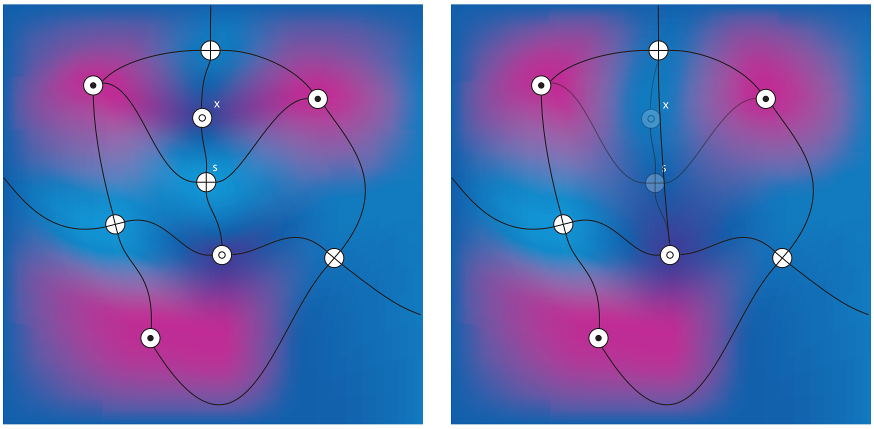


Figure 2.2: A Cancellation of persistence pair (x, s).

2.5 Piecewise Linear Morse Theory

In most applications, we are given function values at the vertices of a mesh. We extend these values to a PL function f on $\mathcal{K}$. Adjacent vertices with the same function value are perturbed so there is an ordering of vertices. Now, since we do not have derivative information as in the smooth case, we use the values of the neighboring vertices to classify a vertex.

We are concerned here with how PL Morse theory classifies critical points, and not so much how one might construct a Morse complex using the PL theory (for PL MS Complex construction on 2-manifolds see [1] and [25] for 3-manifolds). To understand PL point classification, we need the following definitions.

Definition 13 (Star)**.** *The star of a cell α is the set of cells incident to α:*

$$St\ \alpha = \{\beta | \alpha < \beta\}$$

Definition 14 (Link)**.** *The link of a cell is the set of cells on the boundary of the star.*

$$Lk\ \alpha = \partial St\ \alpha - St\ \alpha$$

The star of a cell is not necessarily a cell complex, because the boundary of a cell may not be in the star. The link, however, is always a cell complex.

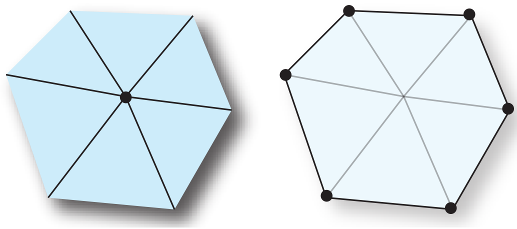


Figure 2.3: Star and link of a vertex. On the left, the vertex and incident edges and triangles (shaded) belong to the star. On the right, the dark ring of edges and vertices belong to the link.

Definition 15 (PL-manifold)**.** *A* PL-manifold *is a cell complex $\mathcal{K}$ such that, for every vertex $v \in \mathcal{K}_0$, $St(v)$ is homeomorphic to a ball.*

Definition 16. *A PL Morse function is a PL function $f : X \rightarrow \mathbb{R}$, over a PL manifold, $\mathcal{K}$, such that $f_0(v) = f_0(w)$ iff $v = w$.*

Definition 17 (Lower Star)**.** *The lower star of a vertex, $St_-\ v$, is the set of cells of the star with vertices of lesser function value:*

$$St_-\ v = \{\beta \in St\ v | \forall v' \in \beta, f(v') < f(v)\}$$

Definition 18 (Lower Link)**.** *The lower link of a vertex, $Lk_- v$, is the set of cells of the link with vertices of lesser function value:*

$$Lk_-\ v = \{\beta \in St\ v | \forall v' \in \beta, f(v') < f(v)\}$$

Due to perturbation, each cell has a unique highest vertex. Therefore, the lower stars of the vertices partition the complex. We will will process the complex by each lower star.

In PL Morse theory, we classify a vertex v by the structure of $Lk_- v$. If the lower link is empty or the entire link, then v is a minimum or a maximum respectively. Otherwise, the link consists of $k+1 \geq 1$ connected components. The vertex is a *regular point* if $k = 0$, and a *k-fold saddle* if $k \geq 1$.

Table 2.1 summarizes the classification of vertices. Intuitively, a vertex is classified as a maximum if all its neighbors are lower, a minimum if all are higher, and a saddle if some are higher and some are lower.

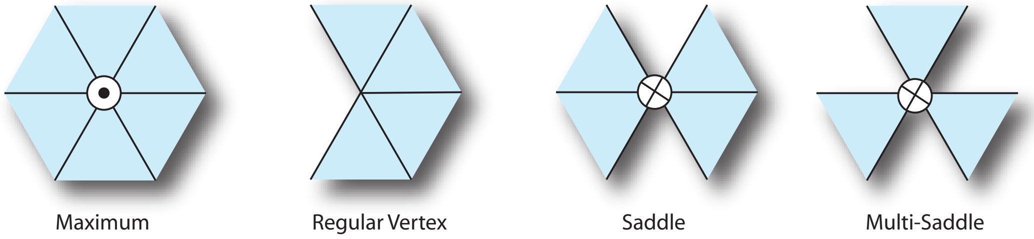


Figure 2.4: Lower Stars of a Maximum, Regular Vertex, Saddle, and Multi-Saddle

Table 2.1: Classification of PL vertices according to structure of the lower link.

Lk_- *components*	*classification*
empty	minimum
closed	maximum
one component	regular point
multiple components	saddle

Chapter 3

Previous Work

This section describes previous work on Morse theory, PL Morse theory, and Discrete Morse theory, leading up to the work in this thesis.

The first developments of what we now call Morse theory was in 1859, when Cayley [2] described the use of contours and integral lines to analyze terrains. In his informal paper, he realized that a saddle ("pass") is incident to two ascending "ridge-lines" and two descending "course-lines," one-manifolds of the MS complex. He also described the use of perturbation to remove degeneracies, the technique used today.

In 1870, Maxwell [20] explicitly described the MS complex as the intersection of the complexes of ride-lines and course-lines (stable and unstable manifolds). He noted the relationship between numbers of critical points of each type, a result closely related to the Critical Point Theorem (§2.1).

In 1925, M. Morse [22] published his first paper, *Relations between the critical points of a real function of n independent variables*, on what is today called *Morse theory*, a branch of *global analysis*. The text used today is [21]. [1] describes the historical developments of Morse theory in greater detail.

Vector Field Topology. Related to Morse Theory is the study of the topology of vector fields. Topology, in this context, refers to the structure of all streamlines in the vector field. Because vector fields are more general than gradient fields (they can contain cycles), the structures are also more complex to generate and the numerical methods even more unstable. The visualization of three-dimensional vector field topology was first presented to the broader scientific visualization community by Helman and Hesselink [17].

Morse Theory on Meshes. Edelsbrunner [8] refined the application of Morse theory on meshes. Bremer constructed a hierarchy of complexes for topological simplification. Ni [26] made the construction of piecewise-linear (PL) Morse complexes easier and more robust. In PL Morse theory, critical points are always vertices. An implementation must deal with special cases such as multi-saddles and crossing paths. Splitting of multi-saddles [8] or the use of "teflon saddles" [26] resolves the problem of multi-saddles. Path crossings can be resolved by subdividing the mesh near crossings or merging paths, which requires extra bookkeeping.

Discrete Morse Theory. Forman [13] introduced discrete Morse theory as a combinatorial analogue of classical Morse theory. Lewiner [19] studied the creation of minimal discrete Morse functions [1], and proved that in arbitrary dimension (check this), constructing a minimal Morse function is MAX-SMP hard [2]. Discrete Morse theory has been applied in some other interesting domains, summarized in [13].

Smooth Morse Functions. Ni [26] showed that smooth Morse functions (functions with few critical points) can be generated through numerical methods. They solve a laplacian system on the mesh which yields a function with a minimal set of critical points. These smooth Morse functions can also be used with our algorithm.

Topological Persistence. The concept of persistence originates in Smale's work [29]. Smale used it as a strategy for cancelling critical points in high dimensional manifolds. In the case of homotopical spheres, the cancellation always reached a minimal configuration, proving the Poincaré conjecture for dimension five and above. Edelsbrunner [10] named this process "topological persistence" and described it in terms of *filtrations* of simplicial complexes, rather than Morse functions.

Simplification of Morse Complexes. Gyulassy [15] described a method for simplification of 3-D Morse-Smale complexes. This method starts with a complex in which every cell is critical and applies a combinatrial simplification, similar in an abstract sense

[1] a Minimal Morse function has the fewest critical points.
[2] an NP-hard problem with no polynomial approximation algorithm

to polygonal mesh simplification, to the Morse-Smale complex. Bremer [1] described a data structure for the storage of a multi-resolution hierarchy of 2-D Morse complexes based on persistence pair cancellation.

Reparameterization. Dong [5] used the MS-Complex for surface reparameterization . Their Morse function was the Laplacian eigenfunction, which distributes extrema evenly over the mesh, making for a desirable quadrangular base mesh. Smoothing was needed to produce a well-shaped quadrangulation from the somewhat jagged MS-Complex.

Shape Analysis. Cazals et al. [3] applied the MS-Complex to the analysis of molecular surfaces. Specifically, they used the MS-Complex to identify surface "knobs" and "depressions" where molecular surfaces dock (join together) to form a complex. They provide a $O(n \log n)$ algorithm for computing the MS-Complex, where n is the size of the mesh. Their algorithm computes the stable and unstable regions using spanning trees, instead of constructing the gradient on lower-stars as does the algorithm in this thesis.

Chapter 4

Discrete Morse Theory

This chapter provides an introduction to Discrete Morse Theory, a formulation of Morse Theory for cell complexes.

4.1 Foundations

Robin Forman introduced a discrete version of Morse theory in [12], which could applied directly to cell complexes. He demonstrated that many of the theorems of the smooth theory also hold under his formulation.

Definition 19. *Given a cell complex, $\mathcal{K}$, a* discrete Morse function *is a function $f : \mathcal{K} \to \mathbb{R}$ such that for every $\alpha \in \mathcal{K}_p$:*

1. $\mid \{\beta \in \mathcal{K}_{p-1} : \beta < \alpha \text{ and } f(\alpha) \leq f(\beta)\} \mid \leq 1$

2. $\mid \{\beta \in \mathcal{K}_{p+1} : \beta > \alpha \text{ and } f(\alpha) \geq f(\beta)\} \mid \leq 1$

A cell $\alpha \in \mathcal{K}$, if both conditions are satisfied with strict equality (i.e. both sets are empty), then α is called a *critical cell* of f. The dimension of α is the index of the critical point. We denote the set of all index p critical points by $\mathcal{M}_p$ and define $m_p = \mid \mathcal{M}_p \mid$. If α is not a critical point, then we say that it is a *regular cell.*

Lemma 4. *For a regular point, $\alpha \in \mathcal{K}_p$, of a discrete Morse function, f, either*

1. $\mid \{\beta \in \mathcal{K}_{p-1} : \beta < \alpha \text{ and } f(\alpha) \leq f(\beta)\} \mid = 1$*, or*

2. $|\{\beta \in \mathcal{K}_{p+1} : \beta > \alpha \text{ and } f(\alpha) \geq f(\beta)\}| = 1.$

The proof of this is given in [12]. This lemma implies that there is a unique pairing of regular points such that each regular point is paired with one of its faces or one of its co-faces. Each such pair is called a discrete vector, and any pairing of cells in this way is called a discrete vector field. The particular pairing defined by the lemma is called the gradient field of f, written V_f.

Definition 20 (Combinatorial Vector Field)**.** *A combinatorial vector field, V, is a function mapping a cell to some cell in its co-boundary:*

$$V\alpha \in \partial^{-1}\alpha$$

Definition 21 (Discrete Gradient Field)**.** *Given a discrete Morse function, f, then the* discrete gradient field *of f is a function $V_f : \mathbb{C}_*(\mathcal{K}) \to C_{*+1}(\mathcal{K})$, such that $\alpha = V_f\beta$ iff $\alpha > \beta$ and $f(\alpha) \leq f(\beta)$.*

A path in the discrete gradient field is called a *gradient path.* This is the combinatorial analog of an *integral line* (Definition 7).

Definition 22. *A gradient path of dimension p is a sequence γ,*

$$\gamma = \alpha_0\alpha_1 \cdots \alpha_n$$

such that, for all $0 \leq k < n$:

1. $\alpha_k \in \mathcal{K}_p$

2. $\alpha_{k+1} < V_f\alpha_k$ *and* $\alpha_{k+1} \neq \alpha_k$

The length of γ, is $|\gamma| = n$.

It is often the case that it is easier to work with gradient fields than the complete Morse function. Thus it is helpful to know when an arbitrary discrete vector field is the gradient field of a Morse function. This is the content of the next theorem, which is also proved in [12].

Theorem 5. *A vector field, V, over a cell complex, $\mathcal{K}$, is a discrete Morse function iff V has no non-trivial cycles.*

A non-trivial cycle in a combinatorial vector field is a V-path where a cell is repeated. This matches our intuition for continuous gradient fields, which are conservative vector fields and therefore do not contain cyclic paths. This property also guarantees our algorithms will terminate.

Definition 23. *The discrete time flow of a discrete gradient field, V, is:*

$$\Phi = 1 + dV + Vd$$

Lemma 6.

$$\chi = \sum_i (-1)^i |\mathcal{K}_i| = \sum_i (-1)^i |\mathcal{M}_i|$$

The proof follows from the fact that introducing a gradient mapping reduces the number of critical simplices by two, and the two are from adjacent dimension. Thus, the χ is preserved.

Definition 24. *A* minimal *Morse function has only two extrema: one maximum and one minimum.*

4.2 Construction

The following theorem shows that we can always create a discrete morse function which coincides with the PL function in terms of critical points.

Theorem 7. *Given an n-dimensional simplicial complex without boundary, $(X, \mathcal{K})$, and a PL function, $f : X \to \mathbb{R}$, there exists a discrete Morse function, $F : K \to \mathbb{R}$ such that for each critical point, $v \in X$ of f with index $\kappa(v) = (k_0, k_1, \ldots, k_d)$, there are exactly k_i critical cells of F in $St_-(v)$ for each $0 \le i \le d$.*

Proof. Rather than constructing the function F we construct its gradient field, V_F. V_F is constructed incrementally over the filtration $\mathcal{K}^0 \subseteq \mathcal{K}^1 \subseteq \cdots \subseteq \mathcal{K}^r = \mathcal{K}$ defined by f. First we define $V^0 = \emptyset$ to be a gradient field over $\mathcal{K}^0$. Clearly this definition satisfies the theorem since $\mathcal{K}^0 = v_0$ and $\kappa(v) = (1, 0, \cdots, 0)$.

We assume that the gradient field V^{i-1} has been constructed over the sub-complex $\mathcal{K}^{i-1}$, such that V^{i-1} defines a discrete Morse function F_{i-1} that satisfies the theorem. Consider the function $G_i = F_{i-1} \mid_{Lk_-(v_i)}$, then G_i is a Morse function on $Lk_-(v_i)$. By theorem .... there exists a minimal Morse function $\tilde{G}_i$ such that $m_p(\tilde{G}_i) = \beta_p(Lk_-(v_i))$ for all $0 \leq p \leq d$. Let $V_{\tilde{G}_i}$ be the gradient field of $\tilde{G}_i$, and let w_i be the global minimum of $\tilde{G}_i$. Now we can define $V^i = V^{i-1} \cup \{(v_i, v_i * w)\} \cup \{(v_i * \alpha, v_i * \beta) : (\alpha, \beta) \in V_{\tilde{G}_i}\}$. It is clear that $V^i - V^{i-1}$ has exactly $\tilde{\beta}_p(Lk_-(v_i))$ unpaired p-simplices. Thus if V^i is the gradient field of a Morse function, F_i, then F_i satisfies the theorem.

To see that V^i is a gradient field we must show that there are no non-trivial cycles. First, note that for $\alpha \in \mathcal{K}^{i-1}$, $V^i(\alpha) \in \mathcal{K}^{i-1}$. Thus any non-trivial cycle must be contained in $St_-(v_i)$. Now, $V_{\tilde{G}_i}$ is the gradient field of a Morse function and so has no non-trivial cycles. Therefore, $\{(v_i * \alpha, v_i * \beta) : (\alpha, \beta) \in V_{\tilde{G}_i}\}$ has non non-trivial cycles. So we only need to show that the vector $(v_i, v_i * w)$ does not create any cycles. This is clear since a gradient path containing v_i must contain $w \in \mathcal{K}^{i-1}$, and therefore cannot contain any other simplices in $St_-(v_i)$. $\square$

4.3 Duality

The following theorem enables us to generate the entire MSC, using only the unstable cells. This is the combinatorial analogue of the unstable manifold (2.1).

Theorem 8. *Given $\alpha \in \mathcal{M}_p$, the dual of the ascending cells of α is the unstable cells of $\alpha^\star$, that is,*

$$a(\alpha)^\star = u(\alpha^\star)$$

Proof. Let $\beta \in a(\alpha)^\star$, then:

$$\Phi^{\star N} \alpha^\star = \beta^\star + c,$$

for some $N \in \mathbb{Z}^+$, and some $c \in C^p(\mathcal{K})$. Therefore,

$$\beta^\star = \Phi^{\star N} \alpha^\star - c,$$

and

$$\begin{aligned} 1 &= \beta^\star \beta \\ &= \Phi^{\star N} \alpha^\star \beta - c\beta \\ &= \Phi^{\star N} \alpha^\star \beta, \end{aligned}$$

since $< c, \beta^\star >= 0$. But this implies that,

$$\Phi^N \beta = \alpha + c',$$

for some $c' \in C_p(\mathcal{K})$. And thus,

$$\Phi^{*N} \alpha^\star = \beta^\star + c'',$$

for some $c'' \in C_p(\mathcal{K}^\star)$. So we have shown that $\beta \in u(\alpha^\star)$, and therefore that $a(\alpha)^\star \subseteq u(\alpha^\star)$. But the exact inverse reasoning proves that $u(\alpha^\star) \subseteq a(\alpha)^\star$, and so we have the theorem. $\square$

Chapter 5

Morse-Smale Complexes on 2-Manifolds

This chapter presents the construction of MS Complexes on piecewise-linear 2-manifold cell complexes. Given a PL function, we construct a discrete gradient field such that the PL critical points correspond to the critical cells of a discrete morse function.

5.1 Gradient Field Construction

We would like to create a gradient field such that for each PL-critical vertex we have a critical simplex within its lower star. We begin by taking a closer look at the lower star.

Theorem 9. *If v is a critical point of index k, the minimal discrete gradient on $St_-\ v$ has k critical cells.*

Proof. On the lower star, we must generate the 1-2 mappings of the discrete gradient. In the case where $Lk_- v$ is a closed loop (we have a maximum), the maximal assignment of 1-2 gradient mappings would result in a cycle. Thus there must be a critical 2-cell. The remaining edge can be paired with v.

If Lk_- is not closed, we can pair every face of St_- with an edge of St_-, leaving one edge unpaired for each component of Lk_-. An unpaired edge can only be paired with

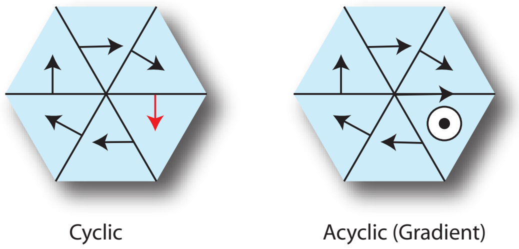


Figure 5.1: The red gradient mapping is removed to break the 1-2 gradient cycle.

the vertex. Thus, if there are n link components, then there will be $n-1$ unpaired edges ($n-1$ saddles). □

This construction corresponds to the process in PL Morse theory of splitting multi-saddles [8], however discrete Morse theory handles it implicitly.

There are many possible gradients. Consider a maximum. Any maximal gradient can be rotated such that any of the faces of $St_-\ v$ can be made critical. In fact, for each lower star in Figure 2.4, there are multiple alternative discrete gradients that are combinatorially valid.

5.1.1 Algorithm

Algorithm 1 GRADIENT(v)

map the vertex to the steepest edge in St_-v
for each component of Lk_-v **do**
    choose the steepest edge $e \in St_-v$ with endpoint in link component
    iterate clockwise and counterclockwise in link component, setting (edge, face) mappings to point toward e
end for

Any choice of an edge e with an endpoint in the link component will suffice to produce an optimal gradient on the lower star (again, optimal in the sense of having the fewest critical cells). However, a poor choice of e can lead to a complex with undesirable geometry (see Figure 5.2). We choose the steepest edge for e.

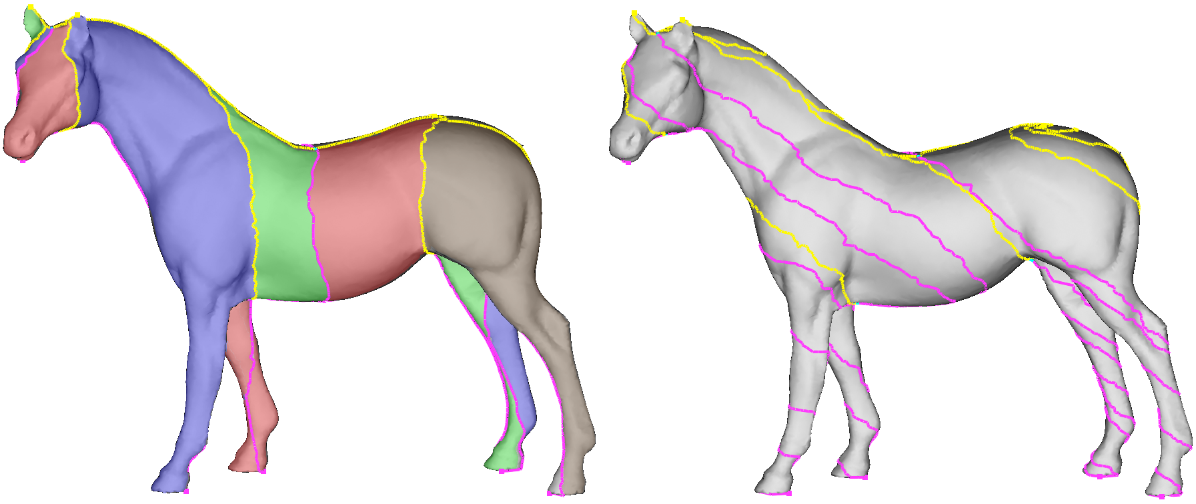


Figure 5.2: A poor gradient heuristic (right)

GRADIENT calculates the discrete gradient on the lower star of a vertex.

Theorem 10. GRADIENT *generates a discrete gradient field.*

Proof. GRADIENT generates a combinatorial vector field. To show this vector field is a discrete gradient field, we must show that there does not exist a cyclic v-path. Let

$$f_{max}(\alpha) = max_{v \in Vt\ \alpha} f(v)$$

The vertex-edge gradient mappings direct a v-path from v to $Lk_-\ v$. Thus all vertex-edge v-paths have non-increasing f_{max} and are acyclic.

Similarly, an edge-face v-path enters $St_-\ v$ or originates at a critical edge and is directed to an edge of $Lk_-\ v$. □

Implementation

GRADIENT computes a gradient field for the large model shown in Figure 5.5 in a few seconds on a 2005 Macintosh, faster than the computer can load the mesh. With the help of the quad-edge data structure [14], this algorithm is straightforward to implement. We store gradient mappings on the directed edges and directed dual-edges, a single bit

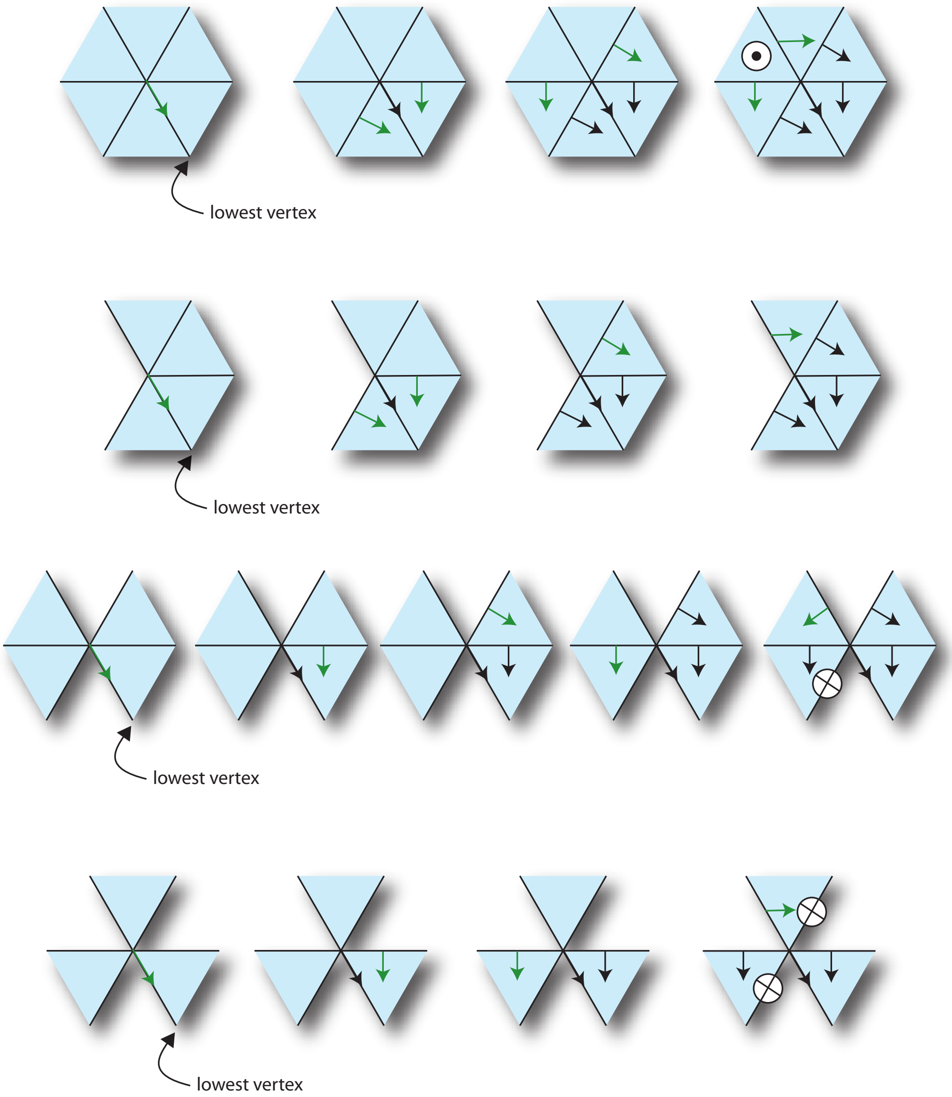


Figure 5.3: The sequence of gradient arrows added by each iteration of GRADIENT.

model	*triangles*	*critical cells*	*gradient computation*	*cancellation*
Bunny	69664	$c_0 = 164, c_1 = 290, c_2 = 128$	0.397120	0.13730
Foot	51690	$c_0 = 167, c_1 = 395, c_2 = 228$	0.302766	0.079841
Hand	654666	$c_0 = 155, c_1 = 308, c_2 = 143$	3.463385	0.747729
Blade	1765388	$c_0 = 4004, c_1 = 8261, c_2 = 3933$	9.488464	21.887555

Figure 5.4: Timings for various models. Cancellation time refers to full cancellation to a minimal number of critical cells. Timings were measured on a 3.40GHz Intel Xeon workstation.

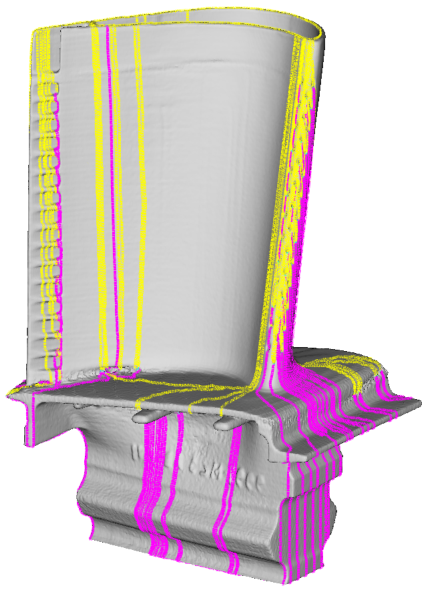


Figure 5.5: A genus-165 turbine blade.

indicating the presence of a gradient pair or inverse gradient pair, respectively. Thus, the primal and dual operations use the same functions. We implemented the algorithm in approximately 300 lines of C. The implementation is given in Appendix A.

5.2 Persistence Simplification

Topological persistence [8], a measure of the importance of a level-set component, is the difference in scalar value of a pair of critical points connected by an arc in the Morse-Smale complex.

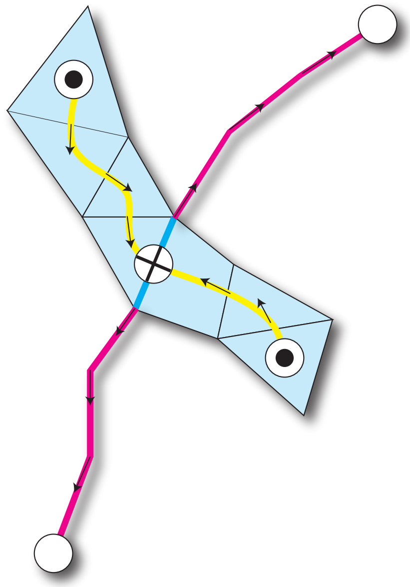


Figure 5.6: Tracing The 1-Manifolds from a saddle (cyan). Descending manifolds are magenta. Ascending manifolds are yellow.

Definition 25 (Persistence)**.** *Let* $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *be a v-path where* α_1 *and* α_n *are critical. Then the persistence of* γ*,* $p(\gamma)$*, is defined as*

$$p(\gamma) = |f_{max}(\gamma) - f_{min}(\gamma)|$$

As suggested by Forman [13], pairs of critical cells connected by a v-path may be cancelled by reversing the gradient along the v-path so long as no cycle is created:

Theorem 11 (9.1 from Forman [13])**.** *Suppose* f *is a discrete Morse function on* M *such that* β_{p+1} *and* α_p *are critical, and there is exactly one gradient path from the boundary of* β *to* α*. Then there is another Morse function* g *on* M *with the same critical simplices except that* α *and* β *are no longer critical. Moreover, the gradient vector field associated with* g *is equal to the gradient vector field associated with* f *except along the unique gradient path from the boundary of* β *to* α*.*

Creation of a cycle is very easy to check. If a saddle's 1-manifolds end at the same minimum or maximum, reversing the gradient along one of the 1-manifolds would create

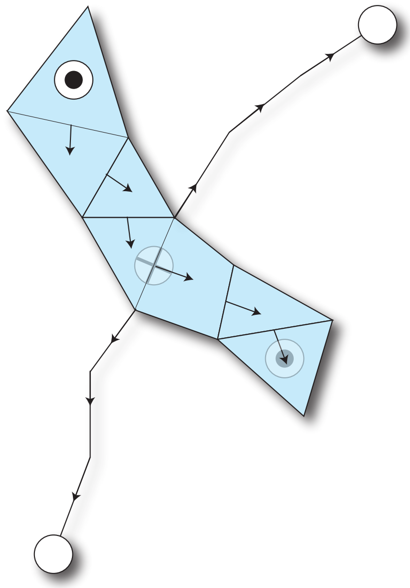


Figure 5.7: Cancelling a pair of critical points from Figure 5.6

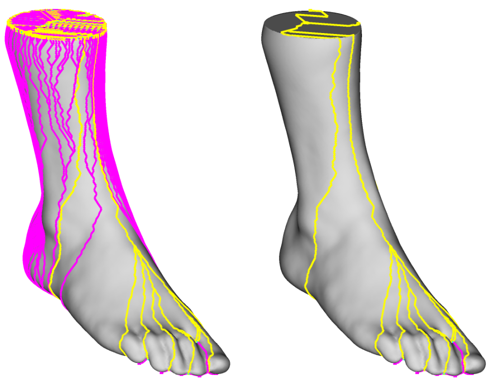


Figure 5.8: Persistence simplification of a foot model. The many critical points on the flat top of the model are simplified by cancellation of low-persistence pairs

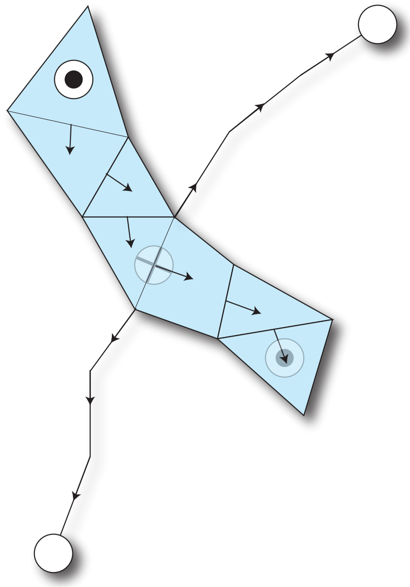


Figure 5.9: Cancelling a pair of critical points from Figure 5.6

a cycle. One could use *path compression* during the construction of the gradient field to accelerate cycle detection.

Algorithm 2 performs persitence-pair cancellation. p_0 is the persistence threshold. Setting p_0 to ∞ yields a minimal morse function in the sense of [19].

Algorithm 2 CANCEL(p_0)

find all unique v-paths originating at saddles
sort v-paths by persistence
for each v-path γ **do**
  if $p(\gamma) < p_0$ **and** cancelling γ would not create a gradient cycle **then**
    reverse gradient pairs along γ
  end if
end for

Figure 5.9 shows the cancellation of a pair of critical cells. Figure 5.8 shows persistence simplification applied to a polygonal model.

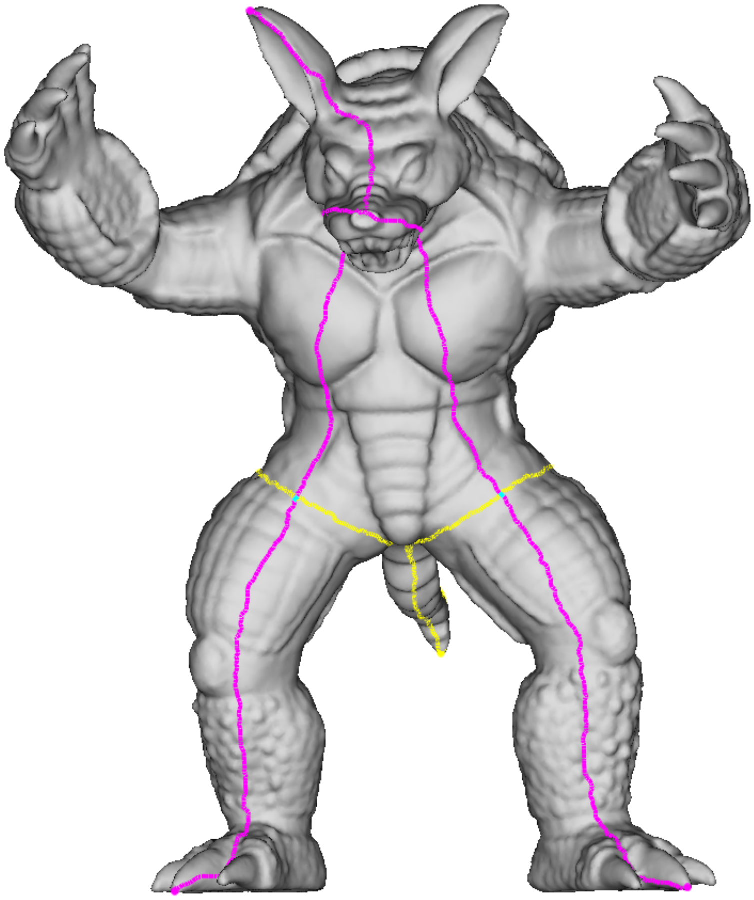


Figure 5.10: Stanford Armadillo Model. Morse function is the Laplacian eigenvector.

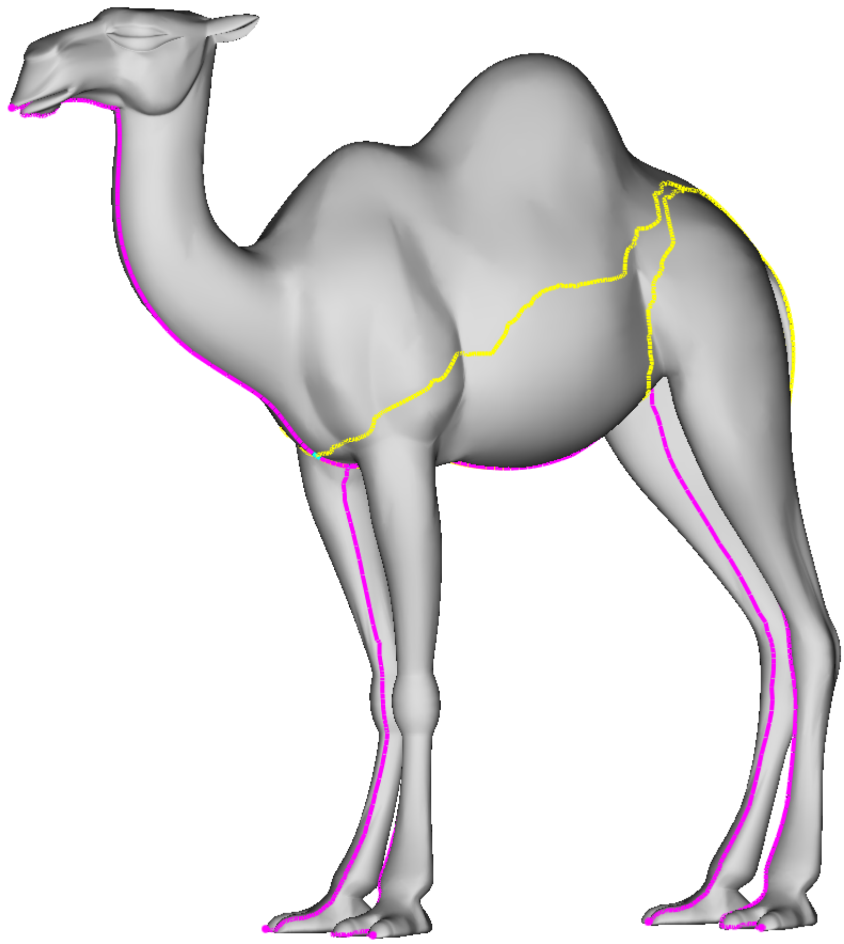


Figure 5.11: Camel Model. Morse function is the Laplacian eigenvector.

Chapter 6

Morse-Smale Complexes on 3-Manifolds

This chapter discusses the computation of Morse-Smale complexes for 3-Manifolds. In particular, we discuss a common format for volumetric scalar fields, rectilinear grid meshes.

6.1 Rectilinear Grid Cell Complex

Given a rectilinear grid cell complex with scalar values at each vertex, we want to store a discrete gradient field. Thus, we will need data at each cell.

A common way to store this information uses pairs of 3-cell (voxel) and face indices, using some canonical ordering of the faces. However this sligtly complicates equality testing of indices since indices and cells are not in a one-to-one correspondance. Instead we adopt the following data structure.

If we subdivide the grid once as we would an octree, then there is one vertex for every cell of the original grid. We use the normal rectilinear indexing on the subdivided grid to index each cell of the original grid.

Let $c = (dual, i, j, k)$ be an *index* of a cell. Define

$$dim(c) = \begin{cases} even(i) + even(j) + even(k) & \text{if } dual = 0 \\ odd(i) + odd(j) + odd(k) & \text{if } dual = 1 \end{cases}$$

Let

$$\begin{aligned} basis_0 &= (0,1,0,0) \\ basis_1 &= (0,0,1,0) \\ basis_2 &= (0,0,0,1) \end{aligned}$$

The following algorithms compute the faces and co-faces of a cell, ∂c and $\partial^{-1}c$ respectively.

Algorithm 3 FACES(idx)

$F \leftarrow \varnothing$
for i = 0 to 2 **do**
  if $odd(idx[i])$ **then**
    $F \leftarrow F \cup idx + basis_i$
    $F \leftarrow F \cup idx - basis_i$
  end if
end for
return F

Algorithm 4 COFACES(idx)

$F \leftarrow \varnothing$
for i = 0 to 2 **do**
  if $even(idx[i])$ **then**
    $F \leftarrow F \cup idx + basis_i$
    $F \leftarrow F \cup idx - basis_i$
  end if
end for
return F

This simple data structure enables efficient MS Complex computation for rectilinear grids, the most common format of volumetric data.

6.2 Layer Graphs

Definition 26 (Hasse Diagram)**.** *The* ***Hasse diagram*** *of a cell complex K is a graph $G = (V, E)$ where $V = K$ and $(\alpha, \beta) \in E$ if and only if $\alpha < \beta$.*

In general, Hasse diagrams may be used to render any partially ordered set (poset).

6.3 Gradient Field Algorithm

Because the topological structure of 3-Manifold lower-stars is more complex than that of 2-Manifold lower-stars, we cannot employ such a simple algorithm to compute a gradient field. Furthermore, because for general cell complexes there is not a one-to-one correspondance between cells of the link and cells of the star, we cannot use the LiftGrad algorithm which works for simplicial complexes, unless we tetrahedralize.

Instead, we use a greedy algorithm, similar to [18]. The algorithm starts with an empty discrete gradient and adds gradient pairs one-by-one, which is equivalent to performing local cancellations on incident cells. We add a pair $(\alpha < \beta)$ if both α and β are critical and adding the pair does not create a cycle in V.

6.3.1 Assignment of Gradient Mappings

If we remove one mapping from a discrete gradient, we still have a discrete gradient (acyclic discrete vector field). Therefore, one can construct any discrete gradient by a sequence of discrete gradients. An algorithm must choose an order to add the mappings.

First, we generate a list of all possible gradient mappings on a lower star, and sort them by a *steepness* heuristic. We have to include mappings that may be sloped up, according to our heuristic, because a minimal gradient may not be possible without them. (why is this the case?)

The correctness of the gradient assignment can be verified by examining the topology of the lower link.

Algorithm 5 CalcGrad(K, *steepness*)

```
L ← ∅ ≪first, we get all possible pairs≫
for all β ∈ K do
  for all α ∈ ∂β do
    L ← L ∪ (α, β)
  end for
end for
sort(L, steepness) ≪sort the pairs with respect to steepness≫
for all (α, β) ∈ L do
  if not Cycle(V, (α, β)) then
    V ← V ∪ (α, β) ≪if the addition of a pair does not create a cycle, add it≫
  end if
end for
return V
```

Algorithm 6 Cycle($V, (\alpha, \beta)$)

```
 ≪determine if we can get back to α from any of the other cells on the boundary of β≫
S ← ∂β − {α}
while S ≠ ∅ do
  let γ ∈ S
  if γ = α then
     ≪we've made our way back to α, so we have a cycle≫
    return true
  end if
  S ← S ∪ (∂Vγ − γ) ≪propagate≫
end while
return false ≪no cycles detected≫
```

Table 6.1: Manifold Generation. β^k is a critical k-cell.

MSC cell	function calls
0-1 manifolds	$\textsc{Unstable}(\beta^1)$
descending surfaces	$\textsc{Unstable}(\beta^2)$
ascending surfaces	$\textsc{Unstable}((\beta^1)^\star)$
1-2 manifolds	$\textsc{Unstable}((\beta^2)) \cap \textsc{Unstable}((\beta^1))^\star$
2-3 manifolds	$\textsc{Unstable}((\beta^2)^\star)$

6.3.2 Simplification of Critical Points

Second, we apply persistence simplification. For 3-manifolds, there is no longer a unique v-path leaving a cell, as there is for 2-manifolds. Thus, we muse use a depth first search to determine critical cells reachable from a given critical cell.

For a critical pair (α, β) to be cancelled, there must be a unique v-path between α and β, so cancellation does not create a cycle. If there is a unique path, the depth first search returns the path.

6.4 Manifold Generation

Unstable generates the *unstable cells* by depth first search of the gradient field. Various uses of Unstable will generate all the manifolds of the morse complex, as summarized in table 6.1.

To generate the 0-1 manifolds, call Unstable on each 1-saddle (edge).

Algorithm 7 UNSTABLE(α)

$Q \leftarrow$ MAKE-QUEUE()

PUSH(Q, s)

$S \leftarrow \varnothing$

while not EMPTY(Q) **do**

  $\beta \leftarrow$ FRONT(Q)

  if $\beta \notin S$ **then**

    $S \leftarrow S \cup \{\beta, V\beta\}$

    for each $f \in \partial V\beta - \beta$ **do**

      PUSH(Q, s)

    end for

  end if

end while

return S

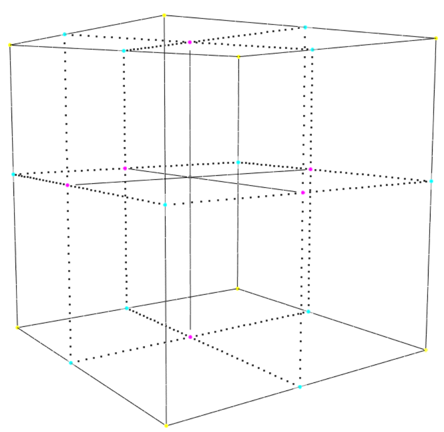


Figure 6.1: MS Complex for a simple function with maximum in the center: $f(x, y, z) = x^2 + y^2 + z^2$. This figure illustrates the structure of the MS-Complex *crystals*. Solid lines are 0-1 and 2-3 manifolds. Dotted lines are 1-2 manifolds.

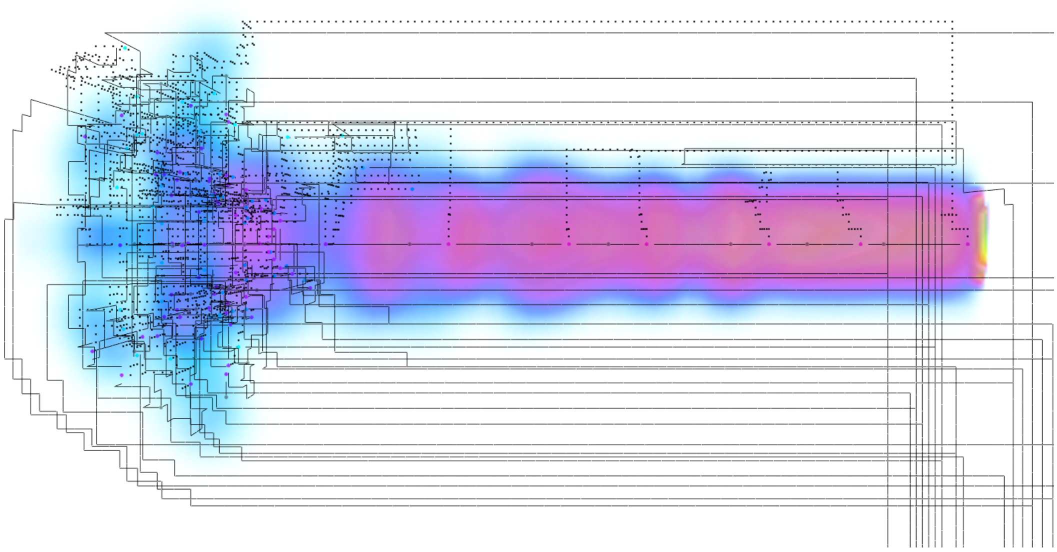


Figure 6.2: Volume rendering of fuel dataset with MS Complex superimposed. Solid lines are 0-1 and 2-3 manifolds. Dotted lines are 1-2 manifolds.

Chapter 7

Conclusion

This thesis has developed algorithms for the creation of discrete gradient fields and discrete Morse-Smale complexes on 2 and 3-manifold cell complexes. By using discrete Morse Theory, we have developed algorithms that are purely combinatorial, easy to implement, and avoid the special cases of previous methods.

Future Research Directions

The computation of MS Complexes on 3-Manifolds is not complete. We have left to future work the problem of showing the gradient fields coincide with PL functions as far as critical points are concerned. Inspection of the data seems to indicate that.

Also, this thesis does not discuss applications of the discrete Morse algorithms in detail. We have assumed these algorithms can replace PL algorithms where they are applied, but actually demonstrating that would be very helpful. Specifically it would be interesting to determine if the approximate alignment of the discrete gradient with the PL gradient causes problems in applications.

Visualizing the entire MS complex on a 3-Manifold would also be interesting. The 2 and 3-cells can get quite convoluted, and muddle the image, if careful visualization techniques are not used.

Analysis of the quality of the heuristics for assigning gradient fields is left to future research. The construction for 2-manifolds favors the primal over the dual by assigning the primal (vertex to edge) mapping to the steepest direction. Thus the primal edges of the complex follow the actual PL gradient, but the dual edges do not. In general this thesis has focused on the combinatorial aspects of the MS Complex as opposed to the geometric. To have a meaningful visualization, it is very important to have a geometrically desirable gradient field, as initial investigation indicated.

Persistence cancellation via gradient reversal, the technique presented here, does not actually simplify the original morse function. This would be a desirable property. Furthermore, after simplification, the resulting gradient field is far less smooth. In the continuous case, cancellation is accomplished by smoothing or locally purturbing the function. In the discrete case, only local purturbations of the gradient field are discussed.

Furthermore, investigation of more efficient methods for persistence simplification of 3-manifold gradient fields than the brute force search presented here would be useful. This might involve storing persistence pairs as the field is created, rather than finding them afterwards.

Appendix A

Source Code

```
void calc_grad(Vertex v) {

  int n = valence(v);
  int i;
  double d = -DBL_MAX;

  if(is_min(v)) return;

  /* sort edges by steepness */
  Edge E[n];
  get_orbit(lath(v), E);
  qsort(E, n, sizeof(Edge), edge_cmp);

  /* now fan out on each edge */
  for(i=0;i<n;++i) {
    Edge e = E[i];

    if(lower(e) and saddle(e)) fanout(e);

  }

  /* set the vertex mapping */
  if(saddle(E[0])) setgrad(E[0]);

}

/* calculate gradient on a component of the lower link
 * steep is steepest edge in component, fan out from it
 */
void fanout(Edge steep) {

  Edge cw, ccw;
```

```
  Vertex v = ORG(steep);

  assert(lower(steep));

  cw = OPREV(steep);
  ccw = ONEXT(steep);

  /* walk left and right */
  while( (cw != ccw)
          and lower(cw)
          and lower(ccw)
          and lowerf(LEFT(cw), v)
          and lowerf(RIGHT(ccw), v)
          and saddle(cw)
          and saddle(ccw)) {
    setgrad(TOR(cw));
    setgrad(ROT(ccw));
    cw = OPREV(cw);
    if(cw == ccw) break;
    ccw = ONEXT(ccw);
  }

  /* walk further left */
  while(lower(cw) and lowerf(LEFT(cw), v) and saddle(cw)) {
    setgrad(TOR(cw));
    cw = OPREV(cw);
  }

  /* walk further right */
  while(lower(ccw) and lowerf(RIGHT(ccw), v) and saddle(ccw)) {
    setgrad(ROT(ccw));
    ccw = ONEXT(ccw);
  }

}
```

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